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Integration Theory of Observables

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Abstract

Representations of abstract observables on a generalised logic are given in terms of bounded vector-valued Borel measures on the real line whose ranges are in the dual space X^* of the Banach space of states X. Each bounded observable is furthermore represented by an element u^* of X^* such that for any proper state $p \in X$, $u^*(p)$ is the expectation value of u when the system is in the state p.

1. *Generalised Quantum Theory*

By *generalised quantum* theory is meant in this paper the list $(\mathscr{L}, \mathscr{S}, \mathcal{O})$ where $\mathscr{L}$ is the proposition system assumed to form an orthocomplemented weakly modular σ-lattice which we call *generalised logic*, $\mathscr{S}$ is the set of (proper) states consisting of all the probability measures on $\mathscr{L}$ and $\mathcal{O}$ is the set of observables consisting of all the σ-homomorphisms on the Borel σ-algebra $\mathscr{B}$ of the real line R into $\mathscr{L}$. Another ingredient to consider is the group $\mathrm{Aut}(\mathscr{S})$ consisting of all the convex automorphisms of $\mathscr{S}$. This contains the symmetry operations of the system under consideration. For definitions see Varadarajan (1968, pp. 105–130).

With such weak conditions on $\mathscr{L}$, the theory is very general and includes both quantum and classical mechanics as special cases. Extra conditions must be imposed on the logic so as to reproduce the conventional quantum formalism in terms of a separable Hilbert space. These, however, are not very clear physically. Restricting ourselves to the generalised case $(\mathscr{L}, \mathscr{S}, \mathcal{O})$, it is necessary to carry out further mathematical developments in order to investigate rigorously questions connected with particles, localisability, dynamics, symmetries and scattering—some of the principal considerations of any physical theory. The main technical problem is that of obtaining useful representations of $\mathscr{S}$, $\mathscr{L}$, $\mathrm{Aut}(\mathscr{S})$ and $\mathcal{O}$ in terms of entities connected with some topological vector space. Our aim in this series is to obtain such representations which, although not as sharp as in the quantum case, are strong enough so as to bring the subject into grips with modern analysis. In the rest of this section are outlined the results obtained showing that the generalised theory is quite manageable mathematically.

In a previous paper (Kronfli, 1970) representations of $\mathscr{S}$ and Aut($\mathscr{S}$) were obtained. Firstly we chose the most important topology, from the physical point of view, on $\mathscr{S}$, namely that defined by the *natural metric*

$$\rho(p,q) = \sup\{|p(a) - q(a)| : a \in \mathscr{L}\} \qquad (p,q \in \mathscr{S})$$

with $\mathscr{S}$ being assumed separating. Let X_1 be the set of all signed measures on $\mathscr{L}$ with finite variations. Defining

$$\|p\| = |p|(I) \qquad (p \in X_1)$$

where $a \to |p|(a)$ is the total variation of p at $a \in \mathscr{L}$ and I is the identity on $\mathscr{L}$ the following were proved

Theorem 1.1

$(X_1, \|\cdot\|)$ *is a real Banach space containing* $\mathscr{S}$ *as a closed convex subset with the norm* $\|\cdot\|$ *inducing the natural metric.*

Theorem 1.2

Each convex automorphism of $\mathscr{S}$ *is represented uniquely by a unit-normed linear one-one operator on* $(X, \|\cdot\|)$ *onto itself, where* X *is the closure of the linear span of* $\mathscr{S}$ *in* $(X_1, \|\cdot\|)$.

These two theorems bear some resemblance to the corresponding ones in quantum theory and for this reason we call X the *Banach space of states.* Problems connected with symmetries, for example, are now easy to handle using operator theory. For an application to abstract scattering see Kronfli (1969).

To complete the picture we consider in this paper the representation theory of the observables $\mathscr{O}$ on the generalised logic $\mathscr{L}$. For quantum logic the observables are represented by projection-valued (spectral) measures on the Borel subsets of R which is equivalent to representing them by the corresponding self-adjoint operators on the Hilbert space of (quantum) states $\mathscr{H}$. It is now the practice to regard each self-adjoint operator on $\mathscr{H}$ as a quantum observable although whether it can actually be observed or not is still a question of debate. Here we shall obtain results rather similar to the quantum case. Each observable u on the generalised logic $\mathscr{L}$ is represented (one-one) by an X^*-valued weakly countably additive Borel measure on R satisfying special boundedness properties similar to those of spectral measures in the quantum case. The set of all such measures is denoted by $\mathscr{M}$. Here X^* is the (Banach) dual of X. The question of regarding each element of $\mathscr{M}$ as an observable is investigated. In fact we give conditions on $\mathscr{O}$ such that each measure in $\mathscr{M}$ represents an observable. When $\mathscr{O}$ satisfies these conditions $\mathscr{M}$ and $\mathscr{O}$ can be identified.

Finally we consider the bounded observables $\mathscr{O}_0$ on $\mathscr{L}$. We shall prove that each $u \in \mathscr{O}_0$ is represented simply by a continuous linear functional u^* on X, i.e. by an element $u^* \in X^*$, with the desirable property that for any state $p \in \mathscr{S}$, $u^*p = \langle p, u^* \rangle$ is the expectation value of u when the system is in the state p. This gives an important physical role to X^* and we can call it the *Banach space of bounded observables.*

2. *Integration Theory of Observables*

The first result is an injection of the logic $\mathscr{L}$ into the dual X^* of X. We shall adopt the following notation: for any linear functional f on X we write

$$\langle p, f \rangle \quad \text{for} \quad f(p) \qquad (p \in X)$$

Proposition 2.1

There exists a natural injection $T: \mathscr{L} \to X^*$ such that

(i) $T(\varnothing) = 0$,
(ii) $\|T(a)\| \leqslant 1 \qquad (a \in \mathscr{L}), \qquad \|T(I)\| = 1$,
(iii) *T is weakly countably additive on $\mathscr{L}$.*

Proof: For each $a \in \mathscr{L}$ consider the map $T(a): p \to p(a)$ on $X \to R$, This is clearly a single-valued real linear functional on X such that $T(\varnothing) = 0$. Furthermore, for any $p \in X$,

$$|\langle p, T(a) \rangle| = |p(a)| \leqslant |p|(a) \leqslant |p|(I) = \|p\|$$

implying that $T(a) \in X^*$ and $\|T(a)\| \leqslant 1$. Also for $p \in \mathscr{S}$, $\langle p, T(I) \rangle = p(I) = 1$ which means that $\|T(I)\| = 1$.

The map $a \to T(a)$ is an injection because $T(a) = T(b)$ implies that $p(a) = p(b)$ for all $p \in X$ and since $\mathscr{S}$ is separating and is contained in X this in turn implies that $a = b$.

Thus T is one-one.

It remains to prove (iii). Let (a_n) be a disjoint sequence in $\mathscr{L}$. Then for any $p \in X$

$$\langle p, T(\bigvee_n a_n) \rangle = p(\bigvee_n a_n) = \sum_n p(a_n) = \sum_n \langle p, T(a_n) \rangle.$$

This completes the proof. ■

Corollary 2.2

The natural injection T of Proposition 2.1 *induces a metric on $\mathscr{L}$ given by*

$$d(a,b) = \|T(a) - T(b)\| \qquad (a, b \in \mathscr{L})$$

Definition 2.3

Let $\mathscr{M}^+$ be the set of all bounded weakly countably additive X^*-valued measures on the Borel σ-algebra $\mathscr{B}$ of the real line. This is a real Banach space when equipped with the norm

$$\|\mu\|_1 = \sup\{\|\mu(A)\| : A \in \mathscr{B}\} \qquad (\mu \in \mathscr{M}^+)$$

For a proof see Dunford & Schwartz (1958). From now on $\|\cdot\|$ without a suffix will denote the norms of X or X^*. Finally we define

$$\mathscr{M} = \{\mu \in \mathscr{M}^+ : \|\mu(A)\| \leqslant 1, \langle p, \mu(A)\rangle \geqslant 0 \qquad (p \in \mathscr{S}, A \in \mathscr{B}),$$
$$\|\mu(R)\| = 1\}$$

The next result is the main representation theorem for $\mathcal{O}$.

Theorem 2.4

There exists a one-one map $u \to \hat{u}$ on $\mathcal{O}$ into $\mathscr{M}$ such that

$$\langle p, \hat{u}(A)\rangle = p(u(A)) \qquad (p \in X, A \in \mathscr{B})$$

Proof: The required mapping $u \to \hat{u}$ is given by $\hat{u} = T o u$ where T is the injection in Proposition 2.1. The rest of the proof is straightforward using Proposition 2.1. ■

Remarks

Let $u \in \mathcal{O}$ and $\hat{u} \in \mathscr{M}$ be the corresponding vector-valued measure as in Theorem 2.4. Then for any proper state $p \in \mathscr{S}$ and $A \in \mathscr{B}$, $\langle p, \hat{u}(A)\rangle$ is the probability of finding the observable u in the Borel set A when the system is in the state p. Note also that $\|\hat{u}\|_1 = 1$. Compare the similarity of the above representation theorem with the corresponding one in quantum theory.

The next result is an important property of the set $\mathscr{M}$.

Theorem 2.6

The set $\mathscr{M}$ is a closed subset of $(\mathscr{M}^+, \|\cdot\|_1)$.

Proof: Let (μ_n) be a sequence in $\mathscr{M}$ which converges to μ in $(\mathscr{M}^+, \|\cdot\|_1)$ Note that $\|\mu_n\|_1 = 1$ and for any $\nu \in \mathscr{M}^+$, $\|\nu(A)\| \leqslant \|\nu\|_1$ for all $A \in \mathscr{B}$. Thus

$$\begin{aligned} |[1 - \|\mu(R)\|]| &= |[\|\mu_n(R)\| - \|\mu(R)\|]| \\ &\leqslant \|\mu_n(R) - \mu(R)\| \\ &\leqslant \|\mu_n - \mu\|_1 \to 0 \qquad (n \to \infty) \end{aligned}$$

Hence $\|\mu(R)\| = 1$. Furthermore, for each $A \in \mathscr{B}$ the sequence $(\|\mu_n(A)\|)$ is bounded by 1 and $\mu_n \to \mu$ implying that $\|\mu(A)\| \leqslant 1$. Similarly $\langle p, \mu(A)\rangle \geqslant 0$ for all $p \in \mathscr{S}$ and $A \in \mathscr{B}$. Thus $\mu \in \mathscr{M}$ and $\mathscr{M}$ is closed. ■

3. *Total and Bounded Observables*

We now come to the question of when can one regard each element of $\mathscr{M}$ as representing an observable. It is obvious that with each observable u there is a probability measure on R corresponding to each state of the system. It sounds reasonable to postulate the converse which roughly says that each map on $\mathscr{S}$ into the class of probability measures on R which satisfies certain properties defines an observable. No philosophical discussion of this is attempted here. We only give the mathematical formulation.

Definition 3.1

Let $\mathscr{P}$ be the set of all probability measures on the real line and let $\mathrm{Hom}(\mathscr{S}, \mathscr{P})$ be the set of all convex homomorphisms on $\mathscr{S}$ into $\mathscr{P}$. Then a set of observables $\mathcal{O}$ on the logic $\mathscr{L}$ is said to be *total* if to each element of $\mathrm{Hom}(\mathscr{S}, \mathscr{P})$ corresponds a unique observable in $\mathcal{O}$.

Theorem 3.2

Let $\mathcal{O}$ be total. Then to each element of $\mathscr{M}$ corresponds an observable.

Proof: Let $\mu \in \mathscr{M}$ and define $P_\mu(E) = \langle p, \mu(E) \rangle$ $(E \in \mathscr{B})$ for each $p \in \mathscr{S}$. By Definition 2.3, $\langle p, \mu(E) \rangle \geqslant 0$ when $p \in \mathscr{S}$. Furthermore,

$$P_\mu(E) = |\langle p, \mu(E) \rangle| \leqslant \|p\| \cdot \|\mu(E)\| \leqslant 1$$

since $p \in \mathscr{S}$ implies $\|p\| = 1$. Also by definition of μ, $P_\mu(\varnothing) = 0$, $P_\mu(R) = 1$. Thus $P_\mu \in \mathscr{P}$. Now consider the map $J_\mu : p \to P_\mu$ of $\mathscr{S}$ into $\mathscr{P}$. It is obvious that $J_\mu \in \mathrm{Hom}(\mathscr{S}, \mathscr{P})$ and since $\mathcal{O}$ is total it defines a unique observable in $\mathcal{O}$. ■

Thus when $\mathcal{O}$ is total we can identify it with $\mathscr{M}$.

Definition 3.3

An observable $u \in \mathcal{O}$ is said to be *bounded* if and only if its support, $\mathrm{supp}(u)$, is a compact subset of R. The set of all bounded observables will be denoted by $\mathcal{O}_0$.

Theorem 3.4

Let $u \to \hat{u}$ be the injection of $\mathcal{O}$ into $\mathscr{M}$ defined in Theorem 2.4. Then the map $u \to u^*$ given by

$$u^* = \int_R x\hat{u}(dx) \qquad (u \in \mathcal{O}_0)$$

maps $\mathcal{O}_0$ into X^* and is such that $\langle p, u^* \rangle$ is the expectation value of u when the system is in the state $p \in \mathscr{S}$.

Proof: Since $\mathrm{supp}(\hat{u})$ is compact, the integral $\int_R x\hat{u}(dx)$ exists and is single-valued in X^*. Thus $u \to u^*$ is a mapping of $\mathcal{O}_0$ into X^*. Now let $p \in X$, then $\langle p, u^* \rangle = \int_R x \langle p, \hat{u}(dx) \rangle$ since $\hat{u}$ is of compact support and hence a regular vector-valued Borel measure, see Dinculeanu (1967). By Theorem

2.4, $\langle p, \hat{u}(A)\rangle = p(u(A))$ $(A \in \mathscr{B})$ and hence for $p \in \mathscr{S}$ $\int_R x\langle p, \hat{u}(dx)\rangle = \int_R x p o u(dx)$ which is the expectation value of u when the system is in the state p. ■

Remarks

With the notation of Theorem 3.4 we have a mapping of $\mathcal{O}_0$ onto the subset

$$\mathscr{X} = \left\{ \int_R x\hat{u}(dx) : u \in \mathcal{O}_0 \right\}$$

of the space X^*. Physically this gives an important role to X^* since each $u^* \in \mathscr{X} \subset X^*$ is a bounded linear functional on X 'representing' a bounded observable u such that for any proper state p, $u^*(p)$ is the expectation value of u in this state.

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